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On the Two-Dimensional Pointwise Dyadic Calculus*

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In this paper we prove the two-dimensional pointwise dyadic differentiability (provided that the distance of the indices is bounded) of the dyadic integral of integrable two-dimensional functions. We also prove some inequality of type (H, L^1) for the maximal operator $\sup_{\mathbf{n} \in \mathbf{N}^2} |d_{\mathbf{n}}(\mathbf{I}f)|$. © 1998 Academic Press

1. INTRODUCTION AND THE MAIN THEOREM

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$, and $I := [0, 1)$. For any set E let E^2 be the cartesian product $E \times E$. Thus $\mathbf{N}^2$ is the set of integral lattice points in the first quadrant and I^2 is the unit square. For $(n_1, n_2) = \mathbf{n} \in \mathbf{N}^2$ set $\bigwedge \mathbf{n} := \max(n_1, n_2)$, $\bigwedge \mathbf{n} := \min(n_1, n_2)$. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j-dimensional Lebesgue measure of any set $E \subset I^j$ by $|E|$. Denote the $L^p(I^j)$ norm of any function f by $\|f\|_p$ $(1 \leqslant p \leqslant \infty)$.

Denote the dyadic expansion of $n \in \mathbf{N}$ and $x \in I$ by $n = \sum_{j=0}^{\infty} n_j 2^j$ and $x = \sum_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = k/2^m$, $k, m \in \mathbf{N}$, choose the expansion which terminates in zeros). n_i, x_i are the ith coordinates of n, x, respectively. Define the dyadic addition $\dot{+}$ as

$$x \dot{+} y = \sum_{j=0}^{\infty} (x_j + y_j \bmod 2)\, 2^{-j-1}.$$

The sets $I_n(x) := \{ y \in I : y_0 = x_0, ..., y_{n-1} \}$ for $x \in I$, $I_n := I_n(0)$ for $n \in \mathbf{P}$, and $I_0(x) := I$ are the dyadic intervals of I.

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Let $(\omega_n, n \in \mathbf{N})$ represent the one-dimensional Walsh–Paley system [6, 10] $(\omega_n(x) = \prod_{k=0}^{\infty} (-1)^{n_k x_k},\ n \in \mathbf{N},\ x \in I)$. Denote by $D_n := \sum_{k=0}^{n-1} \omega_k$ the Walsh–Dirichlet kernels. It is well-known that [6, 10]

$$S_n f = f * D_n,$$

where $*$ represents the dyadic convolution, that is,

$$S_n f(y) = \int_I f(x)\, D_n(y \dot{+} x)\, dx = f * D_n(y),$$

with $(y \in I, n \in \mathbf{P})$ the nth partial sum of Walsh–Fourier series. Moreover ([9, p. 28]),

$$D_{2^n}(x) := \begin{cases} 2^n, & \text{if} \quad x \in I_n, \\ 0, & \text{otherwise}, \end{cases}$$

$$\begin{aligned} D_n(x) &= \omega_n(x) \sum_{k=0}^{\infty} n_k (D_{2^{k+1}}(x) - D_{2^k}(x)) \\ &= \omega_n(x) \sum_{k=0}^{\infty} n_k (-1)^{x_k} D_{2^k}(x), \qquad n \in \mathbf{N}, \quad x \in I. \end{aligned}$$

For each function h defined on I set

$$(d_n h)(t) := \sum_{j=0}^{n-1} 2^{j-1} (h(t) - h(t \dot{+} 2^{-j-1}))$$

for $t \in I$, $n \in \mathbf{P}$. Then h is said to be dyadically differentiable at a point t if $(d_n h)(t)$ converges, as $n \to \infty$, to some finite number $h^{[1]}(t)$.

Butzer and Wagner [2] showed that every Walsh function is dyadically differentiable with

$$\omega_k^{[1]}(t) = k \omega_k(t)$$

for all $t \in I$ and $k \in \mathbf{P}$.

Let W be the function whose Walsh–Fourier coefficients satisfy

$$\hat{W}(k) = \begin{cases} 1 & k = 0 \\ 1/k & k \in \mathbf{P}. \end{cases}$$

The dyadic integral of an $h \in L^1$ is defined to be

$$\mathbf{I} h = h * W,$$

that is,

$$\mathbf{I}h(t)=\int_I h(t\dot{+}s)\; W(s)\; ds$$

for $t\in I$. Since $W\in L^2\subset L^1$ it is clear that $\mathbf{I}h$ is defined for all $h\in L^1$. Schipp [7] obtained the following fundamental theorem of dyadic calculus: if $h\in L^1$ and $\hat{h}(0)=0$, then

$$(\mathbf{I}h)^{[1]}=h$$

a.e. on I.

For each f defined on I^2 set

$$\begin{aligned}(d_{\mathbf{n}}f)(x, y)=\sum_{j_1=0}^{n_1-1}\sum_{j_2=0}^{n_2-1} 2^{j_1+j_2-2}(f(x, y)-f(x, y\dot{+}2^{-j_2-1})\\ -f(x\dot{+}2^{-j_1-1}, y)+f(x\dot{+}2^{-j_1-1}, y\dot{+}2^{-j_2-1}))\end{aligned}$$

for $\mathbf{n}=(n_1, n_2)\in\mathbf{N}^2$ and $(x, y)\in I^2$. Butzer and Engels [1] defined the pointwise two-dimensional dyadic derivative of f to be the limit as $\min(n_1, n_2)\to\infty$ of $(d_{\mathbf{n}}f)(x, y)$, when this limit exists.

Butzer and Engels [1] defined the two-dimensional dyadic integral of $f\in L^1(I^2)$ by $\mathbf{I}f:=f*(W\times W)$, where $*$ denotes the two-dimensional convolution.

Define, for $f\in L^1(I^2)$, the maximal function, the diagonal maximal function, and the hybrid maximal function as

$$f^*(x^1, x^2):=\sup_{(n_1, n_2)\in\mathbf{N}^2}|f*(D_{2^{n_1}}\times D_{2^{n_2}})(x^1, x^2)|,$$

$$f^\circ(x^1, x^2):=\sup_{n\in\mathbf{N}}|f*(D_{2^n}\times D_{2^n})(x^1, x^2)|.$$

and

$$f^{\#}(x^1, x^2):=\sup_{n\in\mathbf{N}} 2^n\left|\int_{I_n(x^1)} f(t, x^2)\; dt\right|.$$

The Hardy spaces H, H°, and $H^{\#}$ are defined as the set of functions f in $L^1(I^2)$ for which the corresponding norms are

$$\|f\|_H:=\|f^*\|_1<\infty,\qquad \|f\|_{H^\circ}:=\|f^\circ\|_1<\infty,\qquad \|f\|_{H^{\#}}:=\|f^{\#}\|_1<\infty.$$

The atomic decomposition is a useful characterization of the Hardy space H°. To demonstrate this let us introduce first the concept of an atom. A bounded measurable function $a: I^2\to\mathbb{C}$ is a H° atom if there exists a dyadic square $I_\tau(x^1)\times I_\tau(x^2)=: J_1\times J_2\subset I^2$ $(\tau\in\mathbf{N},\ \mathbf{x}=(x^1,x^2)\in I^2)$ for which

$$\text{(i)}\quad \int_{J_1\times J_2} a=0,$$
$$\text{(ii)}\quad \|a\|_\infty\leqslant 2^{-2\tau},$$
$$\text{(iii)}\quad \operatorname{supp} a\subset J_1\times J_2.$$

The basic result of the atomic decomposition is stated as follows (see [4]). A function $f\in L^1(I^2)$ is in H° if and only if there exists a sequence $(a_k, k\in\mathbf{N})$ of H° atoms and a sequence $(\mu_k, k\in\mathbf{N})$ of real numbers such that

$$\sum_{k=0}^{\infty}\mu_k a_k=f \qquad \text{a.e. for all}\quad n\in\mathbf{N},$$

and

$$\sum_{k=0}^{\infty}|\mu_k|<\infty.$$

Moreover,

$$\|f\|_{H^\circ}\sim\inf\sum_{k=0}^{\infty}|\mu_k|,$$

where the infimum is taken over all decomposition of the form above.

Schipp and Wade [8] (and later Weisz [12]) proved that if $|f|\in H^\#$ (in particular, if $f\in L\log^+ L(I^2)$) and $\hat f(n_1,n_2)=0$ for $n_1n_2=0$, then

$$d_{\mathbf{n}}(\mathbf{I}f)\to f \qquad \text{as}\quad \min\{n_1,n_2\}\to\infty$$

a.e. on I^2.

The main aim of this paper is to prove.

THEOREM 1. *If $f\in L^1(I^2)$ and $\hat f(n_1,n_2)=0$ for $n_1n_2=0$, then $d_{\mathbf{n}}(\mathbf{I}f)\to f$ a.e., as $n_1,n_2\to\infty$ provided that $|n_1-n_2|\leqslant\beta$, where β is some fixed parameter.*

After the proof of the main theorem—as an application—we prove for the maximal operator $\sup_{|n_1-n_2|\leqslant\beta}|d_{\mathbf{n}}(\mathbf{I}f)|$ that

$$\Big\|\sup_{|n_1-n_1|\leqslant\beta}|d_{\mathbf{n}}(\mathbf{I}f)|\Big\|_1\leqslant c\,\||f|\|_{H^\circ} \qquad (|f|\in H^\circ)$$

(with the same condition as in Theorem 1 on the Fourier coefficients).

1. THE PROOF OF THE MAIN THEOREM

Throughout this paper c will denote a constant which may vary at different occurrences and may depend only on β. Without loss of generality $\beta \in \mathbf{P}$ can be supposed. In order to prove Theorem 1 we need to introduce some more notation. For each $t \in I$ and $n \in \mathbf{N}$ set [7, 9]

$$F_n^1(t) = 2^{-n} \sum_{j=0}^{n} 2^j D_{2^n}(t \dot{+} 2^{-j-1}),$$

$$F_n^2(t) = \sum_{j=0}^{n} 2^j \sum_{l=n}^{\infty} 2^{-l} D_{2^l}(t \dot{+} 2^{-j-1}),$$

and

$$F_n^3(t) = \sum_{j=0}^{\infty} 2^{-j} D_{2^{n+j}}(t).$$

Schipp and Wade in [8] proved that

$$|d_n W| \leqslant 12 \sum_{l=0}^{n-1} (n-l)\, 2^{l-n} F_l^1 + 8 \sum_{i=1}^{3} F_n^i \tag{1}$$

for all $n \in \mathbf{P}$. Set

$$F_n^4 := 12 \sum_{l=0}^{n} (n+1-l)\, 2^{l-n} F_l^1.$$

Then by (1)

$$|d_n W| \leqslant 12(F_n^2 + F_n^3 + F_n^4) =: 12 G_n$$

for each $n \in \mathbf{N}$.

First we prove the following lemmas:

Lemma 2. $\|F_n^i\|_1 \leqslant c$ $(n \in \mathbf{N},\ i = 1, 2, 3)$.

Lemma 3. *Let* $\tau, A \in \mathbf{N}$. *Then*

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^i(t)\, dt \leqslant c \sqrt{2^{\tau-1}} \qquad (i = 1, 2, 3).$$

LEMMA 4. *Let* $\tau, A \in \mathbf{N}$. *Then*

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} |G_n(t)| \, dt \leqslant c \sqrt{2^{\tau - A}} \qquad \text{and} \qquad \|G_A\|_1 \leqslant c.$$

LEMMA 5. *Let* $\tau, A \in \mathbf{N}$. *Then*

$$\int_{I^2 \setminus I^2_\tau} \sup_{\substack{\wedge \mathbf{n} \geqslant A \\ \mathbf{n} \in \mathbf{N}^2}} |G_{n_1}(u) \, G_{n_2}(v)| \, du \, dv \leqslant c \sqrt{2^{\tau - 1}}.$$

Proof of Lemma 2. The proof is very elementary and therefore it is left to the reader. ∎

Proof of Lemma 3. Let $t \in I_\tau \setminus I_{\tau+1}$ (over all in the proof of Lemma 3) and see case $i = 3$.

If $A > \tau$, then $D_{2^n + j}(t) = 0$ for all $n \geqslant A$, $j \in \mathbf{N}$, thus

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^3(t) \, dt = 0.$$

If $A \leqslant \tau$, then

$$\begin{aligned} \sup_{n \geqslant A} F_n^3(t) &\leqslant \sup_{n > \tau} F_n^3(t) + \sup_{\tau \geqslant n \geqslant A} F_n^3(t) \\ &= \sup_{\tau \geqslant n \geqslant A} F_n^3(t) \\ &\leqslant \sup_{\tau \geqslant n \geqslant A} \sum_{j = \tau + 1 - n}^{\infty} 2^{-j} D_{2^n + j}(t) + \sup_{\tau \geqslant n \geqslant A} \sum_{j=0}^{\tau - n} 2^{-j} D_{2^n + j}(t) \\ &= \sup_{\tau \geqslant n \geqslant A} \sum_{j=0}^{\tau - n} 2^{-j} 2^{n+j} \\ &\leqslant c 2^\tau, \end{aligned}$$

thus

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^3(t) \, dt \leqslant c \leqslant c \sqrt{2^{\tau - 1}}.$$

That is, the lemma in the case of $i = 3$ is proved.

Next, see the case $i=1$.

If $A>\tau$, then $j\neq\tau$ implies $D_{2^n}(t\dot{+}2^{-j-1})=0$ $(t\dot{+}2^{-j-1}\notin I_{\tau+1}\supset I_A\supset I_n)$. Thus,

$$F_n^1(t)=2^{-n+\tau}D_{2^n}(t\dot{+}2^{-\tau-1}).$$

For a fixed $n\in\mathbf{N}$ $(n\geqslant A>\tau)$

$$\int_{I_\tau\backslash I_{\tau+1}} 2^{-n+\tau}D_{2^n}(t\dot{+}2^{-\tau-1})=2^{-n+\tau}\int_{I_{\tau+1}} D_{2^n}(t)=2^{-n+\tau}.$$

Consequently,

$$\int_{I_\tau\backslash I_{\tau+1}}\sup_{n\geqslant A} F_n^1(t)\,dt\leqslant\sum_{j=A}^{\infty}\int_{I_\tau\backslash I_{\tau+1}} F_j^1(t)\,dt$$
$$\leqslant\sum_{j=A}^{\infty}2^{-j+\tau}\leqslant c2^{\tau-A}\leqslant c\sqrt{2^{\tau-A}}.$$

If $A\leqslant\tau$, then by the help of the case $A>\tau$ we have

$$\int_{I_\tau\backslash I_{\tau+1}}\sup_{n\geqslant A} F_n^1(t)\,dt\leqslant\int_{I_\tau\backslash I_{\tau+1}}\sup_{n\geqslant\tau+1} F_n^1(t)\,dt+\int_{I_\tau\backslash I_{\tau+1}}\sup_{\tau\geqslant n\geqslant A} F_n^1(t)\,dt$$
$$\leqslant c+\frac{1}{2^{\tau+1}}\sup_{\tau\geqslant n\geqslant A}\sum_{j=0}^{n}2^j2^{-n}2^n\leqslant c\leqslant c\sqrt{2^{\tau-A}}.$$

That is, the Lemma 3 is proved also in the case $i=1$.

The rest is to discuss case $i=2$.

If $A>\tau$, then $j\neq\tau$ implies $D_{2^l}(t\dot{+}2^{-j-1})=0$ $(t\dot{+}2^{-j-1}\notin I_{\tau+1}\supset I_A\supset I_l)$. Consequently,

$$F_n^2(t)=2^\tau\sum_{l=n}^{\infty}2^{-l}D_{2^l}(t\dot{+}2^{-\tau-1}).$$

For a fixed $n\in\mathbb{N}$ $(n\geqslant A>\tau)$

$$\int_{I_\tau\backslash I_{\tau+1}} F_n^2(t)\,dt=2^\tau\sum_{l=n}^{\infty}2^{-l}\int_{I_{\tau+1}} D_{2^l}(t)\,dt\leqslant c2^{\tau-n}.$$

Thus,

$$\int_{I_\tau\backslash I_{\tau+1}}\sup_{n\geqslant A} F_n^2(t)\,dt\leqslant\sum_{n=A}^{\infty}c2^{\tau-n}\leqslant c\sqrt{2^{\tau-A}}.$$

At last let $A \leqslant \tau$. By the help of case $A > \tau$ we have

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^2(t)\, dt \leqslant \int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant \tau+1} F_n^2(t)\, dt + \int_{I_\tau \setminus I_{\tau+1}} \sup_{\tau \geqslant n \geqslant A} F_n^2(t)\, dt$$

$$\leqslant c + \sum_{n=A}^{\tau} \int_{I_\tau \setminus I_{\tau+1}} F_n^2(t)\, dt.$$

By Lemma 2 we have that

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^2(t)\, dt \leqslant c + c(\tau - A + 1) \leqslant c\sqrt{2^{\tau - A}}$$

also in the case $\tau \geqslant A$. The proof of Lemma 3 is complete. ▮

Proof of Lemma 4. By (1) we get

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} |G_n(t)|\, dt \leqslant c \int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant 1} \sum_{l=A}^{n-1} (n-l)\, 2^{l-n} F_l^1$$

$$+ c \int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} \sum_{l=0}^{A-1} (n-l)\, 2^{l-n} F_l^1$$

$$+ c \sum_{i=1}^{3} \int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^i$$

$$=: B_1 + B_2 + B_3.$$

By Lemma 3 it follows that

$$B_1 \leqslant \int_{I_\tau \setminus I_{\tau+1}} \left(\sup_{n \geqslant A} \sum_{l=A}^{n-1} (n-l)\, 2^{l-n} \right) \sup_{n \geqslant A} F_n^1$$

$$\leqslant c \int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geqslant A} F_n^1 \leqslant c\sqrt{2^{\tau - A}}$$

and

$$B_3 \leqslant c\sqrt{2^{\tau - A}}.$$

Since

$$\sup_{n \geqslant A} \sum_{l=0}^{A-1} (n-l)\, 2^{l-n} F_l^1 \leqslant c \sup_{n \geqslant A} \sum_{l=0}^{A-1} 2^{(3/4)(l-n)} F_l^1 \leqslant c \sum_{l=0}^{A-1} 2^{(3/4)(l-A)} F_l^1$$

and by Lemma 3

$$\int_{I_\tau \backslash I_{\tau+1}} F_l^1 \leqslant c\sqrt{2^{\tau-l}},$$

then we have

$$B_2 \leqslant c \sum_{l=0}^{A-1} 2^{(3/4)(l-A)} \sqrt{2^{\tau-l}} \leqslant c\sqrt{2^{\tau-A}}.$$

That is,

$$\int_{I_\tau \backslash I_{\tau+1}} \sup_{n \geqslant A} |G_n(t)| \, dt \leqslant c\sqrt{2^{\tau-A}}.$$

On the other hand, by (1) and Lemma 2 the inequality $\|G_n\|_1 \leqslant c$ follows. The proof of Lemma 4 is complete. ∎

Proof of Lemma 5.

$$I^2 \backslash I_\tau^2 \subset ((I \backslash I_\tau) \times I) \cup (I \times (I \backslash I_\tau)) =: B_1 \cup B_2.$$

By Lemma 4 we have

$$\int_{B_1} \sup_{\substack{\wedge\, \mathbf{n} \geqslant A \\ \mathbf{n} \in \mathbf{N}^2}} |G_{n_1}(u)\, G_{n_2}(v)| \, du \, dv$$

$$\leqslant \sum_{i=0}^{\tau-1} \sum_{n_1=A}^{\infty} \sum_{n_2=n_1-\beta}^{n_1+\beta} \int_{I_i \backslash I_{i+1}} |G_{n_1}(u)| \, du \, \|G_{n_2}\|_1$$

$$\leqslant c \sum_{i=0}^{\tau-1} \sum_{n_1=A}^{\infty} \sqrt{2^{i-n_1}} \leqslant c\sqrt{2^{\tau-A}}.$$

The integral on the set B_2 is similar. The proof of Lemma 5 is complete. ∎

Set for $n < \tau$ $(n, \tau \in \mathbf{N})$

$$F_n^{2,\tau}(t) := \sum_{j=0}^{n} 2^j \sum_{l=\tau}^{\infty} 2^{-l} D_{2^l}(t \dotplus 2^{-j-1})$$

and

$$F_{n,\tau}^2(t) := \sum_{j=0}^{n} 2^j \sum_{l=n}^{\tau-1} 2^{-l} D_{2^l}(t \dotplus 2^{-j-1}),$$

that is, $F_n^2 = F_n^{2,\tau} + F_{n,\tau}^2$.

We prove several lemmas in order to discuss the integral $\int_{I^2\setminus I_\tau^2} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2|\leqslant\beta} |G_{n_1}\times G_{n_2}|$.

LEMMA 6. $\int_{I\setminus I_\tau} \sup_{n<\tau} F_n^{2,\tau}(t)\, dt\leqslant c$.

Proof. Let $t\in I\setminus I_\tau$. Then $t\in I_d\setminus I_{d+1}$ for some $d\in\{0, 1, ..., \tau-1\}$. Then

$$F_n^{2,\tau}(t)=\begin{cases} 2^d\sum_{l=\tau}^{\infty} 2^{-l}D_{2^l}(t\dot{+}2^{-d-1}), & n\geqslant d\\ 0 & n<d.\end{cases}$$

This gives

$$\int_{I_d\setminus I_{d+1}} \sup_{n<\tau} F_n^{2,\tau}(t)\, dt=\int_{I_{d+1}} 2^d\sum_{l=\tau}^{\infty} 2^{-l}D_{2^l}(t)\, dt$$

$$\leqslant 2^d\sum_{l=\tau}^{\infty} 2^{-l}\, \|D_{2^l}\|_1\leqslant c\,\frac{2^d}{2^\tau}.$$

Consequently,

$$\int_{I\setminus I_\tau} \sup_{n<\tau} F_n^{2,\tau}(t)\, dt\leqslant \sum_{d=0}^{\tau-1}\frac{2^d}{2^\tau}\leqslant c.$$

The proof of Lemma 6 is complete. ∎

COROLLARY 7. $\int_{(I\setminus I_\tau)^2} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2|\leqslant\beta} F_{n_1}^{2,\tau}(u)\, F_{n_2}^{2,\tau}(v)\, du\, dv\leqslant c$ $(\tau\in\mathbf{N})$.

(Of course this corollary follows from Lemma 6 without the condition $|n_1-n_2|\leqslant\beta$, but it is useless to omit it since it is supposed everywhere in this paper.) In the proofs of the forthcoming Lemmas 8–10, 12, and 13 we generally write $\sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2|\leqslant\beta}$, however, there may be some more restrictions on the indices n_1, n_2. They are sometimes indicated separately.

LEMMA 8. $\int_{(I\setminus I_\tau)^2} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2|\leqslant\beta} F^2_{n_1,\tau}(u)\, F_{n_2}^{2,\tau}(v)\, du\, dv\leqslant c$ $(\tau\in\mathbf{N})$.

Proof. Let $v\in I\setminus I_\tau$. Then $v\in I_d\setminus I_{d+1}$ for some $d\in\{0, 1, ..., \tau-1\}$. Then

$$F_{n_2}^{2,\tau}(v)=\begin{cases} 2^d\sum_{l=\tau}^{\infty} 2^{-l}D_{2^l}(v\dot{+}2^{-d-1}), & n_2\geqslant d\ (\text{then } n_1\geqslant d-\beta)\\ 0, & n_2<d.\end{cases}\tag{8.0}$$

Let $u \in I \backslash I_\tau$. Then $u \in I_a \backslash I_{a+1}$ for some $a \in \{0, 1, ..., \tau - 1\}$. If $n_1 < a$, then $F^2_{n_1, \tau}(u) = \sum_{j=0}^{n_1} 2^j \sum_{l=n_1}^{\tau-1} 2^{-l} D_{2^l}(u \dot{+} 2^{-j-1}) = \sum_{j=0}^{n_1} 2^j 2^{-n_1} D_{2^{n_1}}(u \dot{+} 2^{-j-1}) = D_{2^{n_1}}(u \dot{+} 2^{-n_1-1}) = D_{2^{n_1}}(u) = 2^{n_1}$. That is

$$F^2_{n_1, \tau}(u) = \begin{cases} 2^a \sum_{l=n_1}^{\tau-1} 2^{-l} D_{2^l}(u \dot{+} 2^{-a-1}), & n_1 \geqslant a \\ 2^{n_1}, & n_1 < a. \end{cases}$$

If $a > n_1 \geqslant n_2 - \beta \geqslant d - \beta$, then

$$\sum_{d=0}^{\tau-1} \sum_{a=\max(0, d-\beta+1)}^{\tau-1} \int_{(I_a \backslash I_{a+1}) \times (I_d \backslash I_{d+1})} \sup_{\substack{n_1 < a \\ |n_1 - n_2| \leqslant \beta}} 2^{n_1} 2^d$$

$$\times \sum_{l=\tau}^{\infty} 2^{-l} D_{2^l}(v \dot{+} 2^{-d-1}) \, du \, dv$$

$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(0, d-\beta+1)}^{\tau-1} \frac{2^a}{2^a} 2^d \sum_{l=\tau}^{\infty} 2^{-l} \| D_{2^l} \|_1$$

$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(0, d-\beta+1)}^{\tau-1} \frac{2^d}{2^\tau}$$

$$\leqslant c \sum_{d=0}^{\tau-1} \frac{2^d}{2^\tau} (\tau - d + \beta) \leqslant c.$$

That is, $n_1 \geqslant a$, $n_2 \geqslant d$ can be supposed. We have two cases:

$$u - 2^{-a-1} \in I_\tau \tag{8.1}$$

and

$$u - 2^{-a-1} \in I_b \backslash I_{b+1} \quad \text{for some} \quad b \in \{a+1, a+2, ..., \tau - 1\}. \tag{8.2}$$

Discuss (8.1)

$F^2_{n_1, \tau}(u) \leqslant c 2^a (\tau - a)$. This gives

$$\sum_{a=0}^{\tau-1} \sum_{d=0}^{\tau-1} \int_{I_\tau(2^{-a-1}) \times (I_d \backslash I_{d+1})} \sup_{\substack{\vee \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} F^2_{n_1, \tau}(u) \, F^{2, \tau}_{n_2}(v) \, du \, dv$$

$$\leqslant \sum_{a=0}^{\tau-1} \sum_{d=0}^{\tau-1} \frac{2^a(\tau - a)}{2^\tau} 2^d \sum_{l=\tau}^{\infty} 2^{-l} \| D_{2^l} \|_1$$

$$\leqslant \sum_{a=0}^{\tau-1} \sum_{d=0}^{\tau-1} \frac{2^a(\tau - a)}{2^\tau} \frac{2^d}{2^\tau} \leqslant c.$$

Discuss case (8.2).

If $n_1 > b$, then for $l \geqslant n_1 > b$ we have $D_{2^l}(u \dot{+} 2^{-a-1}) = 0$. Consequently, $n_1 \leqslant b$ can be supposed. That is, $b \geqslant n_1 \geqslant a$ and $n_2 \geqslant d$. Since $n_1 \geqslant n_2 - \beta \geqslant d - \beta$ we have two subcases:

$$b > a \geqslant d - \beta \tag{8.2.1}$$

and

$$b \geqslant d - \beta > a. \tag{8.2.2}$$

See case (8.2.1).

$$F^2_{n_1, \tau}(u) = 2^a \sum_{l=n_1}^{\tau-1} 2^{-l} D_{2^l}(u \dot{+} 2^{-a-1}) = 2^a \sum_{l=n_1}^{b} 2^{-l} 2^l$$
$$= 2^a (b - n_1 + 1) \leqslant 2^a (b - a + 1).$$

This gives

$$\sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta, 0)}^{\tau-1} \sum_{b=a+1}^{\tau-1} \int_{(I_b(2^{-a-1}) \setminus I_{b+1}(2^{-a-1})) \times (I_d \setminus I_{d+1})} \sup_{\substack{\vee \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} F^2_{n_1, \tau}(u)\, F^{2, \tau}_{n_2}(v)\, du\, dv$$
$$\leqslant \sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta, 0)}^{\tau-1} \sum_{b=a+1}^{\tau-1} \frac{2^a (b - a + 1)}{2^b} 2^d \sum_{l=\tau}^{\infty} 2^{-l} \|D_{2^l}\|_1$$
$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta, 0)}^{\tau-1} \sum_{b=a+1}^{\tau-1} \frac{2^a (b-a)}{2^b} \frac{2^d}{2^\tau}$$
$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta, 0)}^{\tau-1} \frac{2^d}{2^\tau}$$
$$\leqslant c \sum_{d=0}^{\tau-1} \frac{2^d}{2^\tau} (\tau - d + \beta) \leqslant c.$$

See case (8.2.2).

We have $b \geqslant n_1 \geqslant d - \beta$,

$$F^2_{n_1, \tau}(u) = 2^a \sum_{l=n_1}^{\tau-1} 2^{-l} D_{2^l}(u \dot{+} 2^{-a-1}) = 2^a \sum_{l=n_1}^{b} 2^{-l} 2^l$$
$$= 2^a (b - n_1 + 1) \leqslant 2^a (b - d + \beta + 1).$$

Consequently,

$$\sum_{a=0}^{\tau-1} \sum_{d=a+\beta+1}^{\tau-1} \sum_{b=d-\beta}^{\tau-1} \int_{(I_b(2^{-a-1})\setminus I_{b+1}(2^{-a-1}))\times(I_d\setminus I_{d+1})} \sup_{\substack{\vee\, \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} F^2_{n_1,\tau}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv$$

$$\leqslant \sum_{a=0}^{\tau-1} \sum_{d=a+\beta+1}^{\tau-1} \sum_{b=d-\beta}^{\tau-1} \frac{2^a(b-d+\beta+1)}{2^b}\frac{2^d}{2^\tau}$$

$$\leqslant c \sum_{a=0}^{\tau-1} \sum_{d=a+\beta+1}^{\tau-1} \frac{2^a}{2^\tau} \leqslant c \sum_{a=0}^{\tau-1} \frac{2^a}{2^\tau}(\tau-a) \leqslant c.$$

We have discussed all the cases, and the proof of Lemma 8 is complete. ∎

LEMMA 9. $\int_{(I\setminus I_\tau)^2} \sup_{\vee\, \mathbf{n}<\tau,\, |n_1-n_2|\leqslant\beta} F^4_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv \leqslant c \ (\tau\in\mathbf{N})$.

Proof. The first step in the proof is the same as in the proof of Lemma 8.

Let $v\in I\setminus I_\tau$. Then $v\in I_d\setminus I_{d+1}$ for some $d\in\{0,1,...,\tau-1\}$. Then for $F^{2,\tau}_{n_2}(v)$ we have the same formula as in (8.0). Let $u\in I\setminus I_\tau$. Then $u\in I_a\setminus I_{a+1}$ for some $a\in\{0,1,...,\tau-1\}$. If $n_1<a$, then for $l\leqslant n_1<a$ $F^1_l(u)=2^{-l}\sum_{j=0}^{l} 2^j D_{2^l}(u\dot{+}2^{-j-1})=2^{-l}\sum_{j=0}^{l} 2^j D_{2^l}(2^{-j-1})=2^l$, thus $F^4_{n_1}(u)\leqslant c2^{n_1}$. In this case the situation is the same as it was in the proof of Lemma 8.

That is, $n_1\geqslant a$ can be supposed.

We have two cases:

$$u-2^{-a-1}\in I_\tau \tag{9.1}$$

and

$$u-2^{-a-1}\in I_b\setminus I_{b+1} \qquad \text{for some} \quad b\in\{a+1,a+2,...,\tau-1\}. \tag{9.2}$$

Investigate (9.1).

$$F^4_{n_1}(u)=\sum_{l=0}^{a-1}(n_1+1-l)\,2^{l-n_1}F^1_l(u)$$

$$+\sum_{l=a}^{n_1}(n_1+1-l)\,2^{l-n_1}2^a2^{-l}D_{2^l}(u\dot{+}2^{-a-1})$$

$$\leqslant c((n_1+2-a)\,2^{2a-n_1}+2^a)\leqslant c2^a,$$

$$\sum_{a=0}^{\tau-1}\sum_{d=0}^{\tau-1}\int_{I_\tau(2^{-a-1})\times(I_d\setminus I_{d+1})} \sup_{\substack{\vee\, \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv$$

$$\leqslant c\sum_{a=0}^{\tau-1}\sum_{d=0}^{\tau-1}\frac{2^a}{2^\tau}2^d\sum_{l=\tau}^{\infty}2^{-l}\,\|D_{2^l}\|_1\leqslant c\sum_{a=0}^{\tau-1}\sum_{d=0}^{\tau-1}\frac{2^a}{2^\tau}\frac{2^d}{2^\tau}\leqslant c.$$

In case (9.2) we have three subcases:

$$n_1 < b, \qquad d-\beta < a \leqslant n_1, \tag{9.2.1}$$

$$n_1 < b_1, \qquad a \leqslant d-\beta \leqslant n_1, \tag{9.2.2}$$

$$a, d-\beta, b \leqslant n_1. \tag{9.2.3}$$

See case (9.2.1). As in case (9.1) we have $F^4_{n_1}(u) \leqslant c2^a$. This gives

$$\sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta+1,0)}^{\tau-1} \sum_{b=a+1}^{\tau-1} \int_{(I_b(2^{-a-1})\backslash I_{b+1}(2^{-a-1}))\times(I_d\backslash I_{d+1})} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv$$

$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta+1,0)}^{\tau-1} \sum_{b=a+1}^{\tau-1} \frac{2^a}{2^b} 2^d \sum_{l=\tau}^{\infty} 2^{-l} \|D_{2^l}\|_1$$

$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta+1,0)}^{\tau-1} \sum_{b=a+1}^{\tau-1} \frac{2^a}{2^b} \frac{2^d}{2^\tau}$$

$$\leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(d-\beta+1,0)}^{\tau-1} \frac{2^d}{2^\tau} \leqslant c \sum_{d=0}^{\tau-1} \frac{2^d}{2^\tau} (\tau-d+\beta) \leqslant c.$$

Discuss case (9.2.2). Like in case (9.2.1), we have $F^4_{n_1}(u) \leqslant c2^a$,

$$\sum_{a=0}^{\tau-1} \sum_{d=a+\beta}^{\tau-1} \sum_{b=d-\beta+1}^{\tau-1} \int_{(I_b(2^{-a-1})\backslash I_{b+1}(2^{-a-1}))\times(I_d\backslash I_{d+1})} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv$$

$$\leqslant c \sum_{a=0}^{\tau-1} \sum_{d=a+\beta}^{\tau-1} \sum_{b=d-\beta+1}^{\tau-1} \frac{2^a}{2^b} \frac{2^d}{2^\tau} \leqslant c \sum_{a=0}^{\tau-1} \frac{2^a}{2^\tau} (\tau-a) \leqslant c.$$

The rest is to investigate case (9.2.3). Then,

$$F^4_{n_1}(u) = \sum_{l=0}^{a-1} (n_1+1-l)\, 2^{l-n_1} F^1_l(u)$$

$$+ \sum_{l=a}^{b} (n_1+1-l)\, 2^{l-n_1} 2^a 2^{-l} D_{2^l}(u \dotplus 2^{-a-1})$$

$$\leqslant c(n_1+2-a)\, 2^{2a-n_1} + 2^a \sum_{l=a}^{b} (n_1+1-l)\, 2^{l-n_1}$$

$$\leqslant c2^a(n_1+1-b)\, 2^{b-n_1}. \tag{9.2.3.0}$$

Since $n_1 \geqslant d-\beta$ (otherwise, $F^{2,\tau}_{n_2}(v)=0$), we have three subcases:

$$a<b\leqslant d-\beta\leqslant n_1, \tag{9.2.3.1}$$

$$a\leqslant d-\beta<b\leqslant n_1, \tag{9.2.3.2}$$

$$d-\beta<a<b\leqslant n_1. \tag{9.2.3.3}$$

Investigate (9.2.3.1).

$$F^4_{n_1}(u)\leqslant c2^a(d-b-\beta+1)\,2^{b-d+\beta},$$

$$\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\sum_{d=b+\beta}^{\tau-1}\int_{(I_b(2^{-a-1})\setminus I_{b+1}(2^{-a-1}))\times(I_d\setminus I_{d+1})}\sup_{\substack{\vee\,\mathbf{n}<\tau\\ |n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\,F^{2,\tau}_{n_2}(v)\,du\,dv$$

$$\leqslant c\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\sum_{d=b+\beta}^{\tau-1}\frac{2^a(d-b-\beta+1)\,2^{b-d+\beta}}{2^b}\frac{2^d}{2^\tau}$$

$$\leqslant c\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\frac{2^a}{2^\tau}(\tau-b)^2\leqslant c\sum_{a=0}^{\tau-1}\frac{2^a}{2^\tau}(\tau-a)^3\leqslant c.$$

Discuss (9.2.3.2). Then (see (9.2.3.0)) we have $F^4_{n_1}(u)\leqslant c2^a$. Thus,

$$\sum_{a=0}^{\tau-1}\sum_{d=a+\beta}^{\tau-1}\sum_{b=d-\beta+1}^{\tau-1}\int_{(I_b(2^{-a-1})\setminus I_{b+1}(2^{-a-1}))\times(I_d\setminus I_{d+1})}\sup_{\substack{\vee\,\mathbf{n}<\tau\\ |n_1-n_2|\leqslant\beta}} F^2_{n_1}(u)\,F^{2,\tau}_{n_2}(v)\,du\,dv$$

$$\leqslant c\sum_{a=0}^{\tau-1}\sum_{d=a+\beta}^{\tau-1}\sum_{b=d-\beta+1}^{\tau-1}\frac{2^a}{2^b}\frac{2^d}{2^\tau}$$

$$\leqslant c\sum_{a=0}^{\tau-1}\sum_{d=a+\beta}^{\tau-1}\frac{2^a}{2^\tau}\leqslant c\sum_{a=0}^{\tau-1}\frac{2^a}{2^\tau}(\tau-a)\leqslant c.$$

Finally, see case (9.2.3.3). As in case (9.2.3.2) we have $F^4_{n_1}(u)\leqslant c2^a$ (see (9.2.3.0)).

$$\sum_{d=0}^{\tau-1}\sum_{a=\max(0,\,d-\beta+1)}^{\tau-1}\sum_{b=a+1}^{\tau-1}\int_{(I_b(2^{-a-1})\setminus I_{b+1}(2^{-a-1}))\times(I_d\setminus I_{d+1})}\sup_{\substack{\vee\,\mathbf{n}<\tau\\ |n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\,F^{2,\tau}_{n_2}(v)\,du\,dv$$

$$\leqslant c\sum_{d=0}^{\tau-1}\sum_{a=\max(0,\,d-\beta+1)}^{\tau-1}\sum_{b=a+1}^{\tau-1}\frac{2^a}{2^b}\frac{2^d}{2^\tau}$$

$$\leqslant c\sum_{d=0}^{\tau-1}\sum_{a=\max(0,\,d-\beta+1)}^{\tau-1}\frac{2^d}{2^\tau}\leqslant c\sum_{d=0}^{\tau-1}\frac{2^d}{2^\tau}(\tau-d+\beta)\leqslant c.$$

All the cases discussed, the proof of Lemma 9 is complete. ∎

Set for $n<\tau$

$$F^3_{n,\tau} := \sum_{j=0}^{\tau-n-1} 2^{-j} D_{2^{n+j}}, \qquad F^{3,\tau}_n := \sum_{j=\tau-n}^{\infty} 2^{-j} D_{2^{n+j}}.$$

Then, $F^3_n = F^3_{n,\tau} + F^{3,\tau}_n$.

If $u \in I \backslash I_\tau$, then

$$F^3_n(u) = \begin{cases} F^3_{n,\tau}(u), & n<\tau, \\ 0, & n \geqslant \tau. \end{cases} \tag{10}$$

LEMMA 10. $\int_{(I\backslash I_\tau)^2} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2| \leqslant \beta} F^3_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv \leqslant c \ (\tau \in \mathbf{N})$.

Proof. Let $u \in I \backslash I_\tau$. Then $u \in I_a \backslash I_{a+1}$ for some $a \in \{0, 1, ..., \tau-1\}$. This gives

$$F^3_{n_1}(u) = \begin{cases} \sum_{j=0}^{a-n_1} 2^{n_1} = (a-n_1+1)\, 2^{n_1} \leqslant c 2^a, & n_1 \leqslant a, \\ 0, & n_1 > a. \end{cases}$$

Let $v \in I \backslash I_\tau$. Then $v \in I_d \backslash I_{d+1}$ for some $d \in \{0, 1, ..., \tau-1\}$. This implies that for $F^{2,\tau}_{n_2}(v)$ we have the same formula as in (8.0). That is, $a \geqslant n_1 \geqslant n_2 - \beta \geqslant d - \beta$ can be supposed.

$$\int_{(I\backslash I_\tau)^2} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2| \leqslant \beta}} F^3_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv \leqslant c \sum_{d=0}^{\tau-1} \sum_{a=\max(0,\, d-\beta)}^{\tau-1} \frac{2^a}{2^a} \frac{2^d}{2^\tau}$$

$$\leqslant c \sum_{d=0}^{\tau-1} \frac{2^d}{2^\tau} (\tau - d + \beta) \leqslant c.$$

This completes the proof of Lemma 10. ∎

Corollary 7 and Lemmas 8, 9, and 10 give

COROLLARY 11. $\int_{(I\backslash I_\tau)^2} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2| \leqslant \beta} F^i_{n_1}(u)\, F^{2,\tau}_{n_2}(v)\, du\, dv \leqslant c$ *for* $i = 2, 3, 4$ $(\tau \in \mathbf{N})$.

Next, we prove the boundedness of integral $\int_{(I\backslash I_\tau) \times I_\tau} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2| \leqslant \beta} F^i_{n_1}(u)\, F^j_{n_2}(v)\, du\, dv$ for some i and j.

LEMMA 12. $\int_{(I\backslash I_\tau) \times I_\tau} \sup_{\vee \mathbf{n}<\tau,\, |n_1-n_2| \leqslant \beta} F^i_{n_1}(u)\, F^j_{n_2}(v)\, du\, dv \leqslant c,$ *where* $i = 2, 3$ *and* $j = 2, 3, 4$.

Proof. Let $v \in I_\tau$, $v \neq 0$. Then $v \in I_d \backslash I_{d+1}$ for some $d \geqslant \tau$. As in the beginning of the proof of Lemma 10 we have $F^3_{n_2}(v) = (d - n_2 + 1)\, 2^{n_2}$.

On the other hand, $F^{2,\tau}_{n_2}(v)=\sum_{j=0}^{n_2} 2^j \sum_{l=\tau}^{\infty} 2^{-l} D_{2^l}(v \dot{+} 2^{-j-1})=0$ $(j\leqslant n_2<\tau)$, $F^2_{n_2,\tau}(v)=\sum_{j=0}^{n_2} 2^j \sum_{l=n_2}^{\tau-1} 2^{-l} D_{2^l}(v\dot{+}2^{-j-1})=2^{n_2}2^{-n_2} D_{2^{n_2}}(2^{-n_2-1})$ $=2^{n_2}$, and for $l\leqslant n_2<\tau$ we have $F^1_l(v)=2^{-l}\sum_{j=0}^{l} 2^j D_{2^l}(v\dot{+}2^{-j-1})=2^{-l}\sum_{j=0}^{l} 2^j D_{2^l}(2^{-j-1})=2^l$. Thus, $F^4_{n_2}(v)\leqslant c2^{n_2}$. This implies that $j=3$ can be supposed.

Let $u\in I\backslash I_\tau$. Then $u\in I_a\backslash I_{a+1}$ for some $a\in\{0, 1, ..., \tau-1\}$.

Set $i=3$. Then

$$F^3_{n_1}(u)=\begin{cases}2^{n_1}(a-n_1+1), & n_1\leqslant a\\ 0, & n_1>a.\end{cases}$$

That is,

$$\begin{aligned}&\int_{(I\backslash I_\tau)\times I_\tau} \sup_{\substack{\vee\,\mathbf{n}<\tau\\ |n_1-n_2|\leqslant\beta}} F^3_{n_1}(u)\, F^3_{n_2}(v)\, du\, dv\\
&\leqslant \sum_{a=0}^{\tau-1}\sum_{d=\tau}^{\infty} \sup_{\substack{n_1\leqslant a,\, n_2<\tau\\ |n_1-n_2|\leqslant\beta}} \frac{2^{n_1}(a-n_1+1)}{2^a}\frac{2^{n_2}(d-n_2+1)}{2^d}\\
&\leqslant c\sum_{a=0}^{\tau-1}\sum_{d=\tau}^{\infty}\frac{2^a2^{a+\beta}(d-a)}{2^{a+d}}\leqslant c\sum_{d=\tau}^{\infty}\frac{2^\tau}{2^d}(d-\tau)\leqslant c.\end{aligned}$$

Next, set $i=2$. By Lemma 6 we have

$$\begin{aligned}&\int_{(I\backslash I_\tau)\times I_\tau} \sup_{\substack{\vee\,\mathbf{n}<\tau\\ |n_1-n_2|\leqslant\beta}} F^{2,\tau}_{n_1}(u)\, F^3_{n_2}(v)\, du\, dv\\
&\leqslant c\int_{I_\tau}\sup_{n_2<\tau} F^2_{n_2}(v)\, dv\leqslant c\sum_{d=\tau}^{\infty}\frac{1}{2^d}\sup_{n_2<\tau} 2^{n_2}(d-n_2+1)\\
&\leqslant c\sum_{d=\tau}^{\infty}\frac{2^\tau}{2^d}(d-\tau+1)\leqslant c.\end{aligned}$$

At last we prove the boundedness of $\int_{(I\backslash I_\tau)\times I_\tau}\sup_{\vee\,\mathbf{n}<\tau,\, |n_1-n_2|\leqslant\beta} F^2_{n_1,\tau}(u)$ $F^3_{n_2}(v)\, du\, dv$.

Let $u\in I\backslash I_\tau$. Then $u\in I_a\backslash I_{a+1}$ for some $a\in\{0, 1, ..., \tau-1\}$.

$$F^2_{n_1,\tau}(u)=\begin{cases}2^a\displaystyle\sum_{l=n_1}^{\tau-1} 2^{-l}D_{2^l}(u\dot{+}2^{-a-1}), & n_1\geqslant a\\ 2^{n_1}, & n_1<a.\end{cases}$$

If $n_1 < a$, then

$$\sum_{a=0}^{\tau-1} \sum_{d=\tau}^{\infty} \int_{(I_a \setminus I_{a+1}) \times (I_d \setminus I_{d+1})} \sup_{\substack{n_1 < a \\ |n_1 - n_2| \leqslant \beta}} F^2_{n_1, \tau}(u)\, F^3_{n_2}(v)\, du\, dv$$

$$\leqslant \sum_{a=0}^{\tau-1} \sum_{d=\tau}^{\infty} \sup_{\substack{n_1 < a \\ |n_1 - n_2| \leqslant \beta}} \frac{2^{n_1}(d - n_2 + 1)\, 2^{n_2}}{2^a 2^d}$$

$$\leqslant \sum_{a=0}^{\tau-1} \sum_{d=\tau}^{\infty} \frac{2^a 2^{a+\beta}(d - a + 1 + \beta)}{2^a 2^d} \leqslant c.$$

That is, $n_1 \geqslant a$ can be supposed.

We have two cases:

$$u - 2^{-a-1} \in I_\tau \tag{12.1}$$

and

$$u - 2^{-a-1} \in I_b \setminus I_{b+1} \qquad \text{for some} \quad b \in \{a+1, a+2, ..., \tau - 1\}. \tag{12.2}$$

In case (12.1), $F^2_{n_1, \tau}(u) \leqslant 2^a(\tau - a)$. Thus,

$$\sum_{a=0}^{\tau-1} \int_{I_\tau(2^{-a-1}) \times I_\tau} \sup_{\substack{\vee\, \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} F^2_{n_1, \tau}(u)\, F^3_{n_2}(v)\, du\, dv$$

$$\sum_{a=0}^{\tau-1} \sum_{d=\tau}^{\infty} \frac{2^a(\tau - a)}{2^\tau}\,(d - \tau)\,\frac{2^\tau}{2^d} \leqslant c.$$

In case (12.2), $F^2_{n_1, \tau}(u)$ can be different from zero only when $a \leqslant n_1 \leqslant b$. This gives

$$\sum_{d=\tau}^{\infty} \sum_{a=0}^{\tau-1} \sum_{b=a+1}^{\tau-1} \int_{(I_b(2^{-a-1}) \setminus I_{b+1}(2^{-a-1})) \times (I_d \setminus I_{d+1})} \sup_{\substack{\vee\, \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} F^2_{n_1, \tau}(u)\, F^3_{n_2}(v)\, du\, dv$$

$$\leqslant \sum_{d=\tau}^{\infty} \sum_{a=0}^{\tau-1} \sum_{b=a+1}^{\tau-1} \frac{1}{2^{b+d}} \sup_{\substack{n_1, n_2 < \tau,\, |n_1 - n_2| \leqslant \beta \\ a \leqslant n_1 \leqslant b}} 2^a(b - n_1 + 1)\, 2^{n_2}(d - n_2 + 1)$$

$$\leqslant c \sum_{d=\tau}^{\infty} \sum_{a=0}^{\tau-1} \sum_{b=a+1}^{\tau-1} \frac{1}{2^{b+d}} 2^{a+b+\beta}(d-b+1)$$

$$\leqslant c \sum_{d=\tau}^{\infty} \sum_{a=0}^{\tau-1} \frac{2^a}{2^d} (d-a)^2 \leqslant c \sum_{d=\tau}^{\infty} \frac{2^\tau}{2^d} (d-\tau)^2 \leqslant c.$$

The proof of Lemma 12 is complete. ∎

LEMMA 13. $\int_{(I \setminus I_\tau) \times I_\tau} \sup_{\vee \mathbf{n} < \tau, |n_1 - n_2| \leqslant \beta} F^4_{n_1}(u) F^j_{n_2}(v) \, du \, dv \leqslant c$, *where* $j = 2, 3, 4$.

Proof. In the same way as in Lemma 12, $j=3$ can be supposed.

Let $v \in I_\tau$, $v \neq 0$. Then $v \in I_d \setminus I_{d+1}$ for some $d \geqslant \tau$. As in the proof of Lemma 12 we have $F^3_{n_2}(v) = (d - n_2 + 1) \, 2^{n_2}$.

Let $u \in I \setminus I_\tau$. Then $u \in I_a \setminus I_{a+1}$ for some $a \in \{0, 1, ..., \tau - 1\}$. Then for $n_1 < a$ we have $F^4_{n_1}(u) \leqslant c 2^{n_1}$ (see the beginning of the proof of Lemma 9). This case can be treated in the same way as in the proof of Lemma 12 (for $F^2_{n_1, \tau}(u)$). Consequently, $n_1 \geqslant a$ can be supposed.

We have cases

$$u - 2^{-a-1} \in I_\tau \tag{13.1}$$

and

$$u - 2^{-a-1} \in I_b \setminus I_{b+1} \qquad \text{for some} \quad b \in \{a+1, a+2, ..., \tau - 1\}. \tag{13.2}$$

Investigate (13.1). Then we already have (in the proof of Lemma 9, investigation of (9.1)) $F^4_{n_1}(u) \leqslant c 2^a$.

$$\sum_{a=0}^{\tau-1} \sum_{d-\tau}^{\infty} \int_{I_\tau(2^{-a-1}) \times (I_d \setminus I_{d+1})} \sup_{\substack{\vee \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} F^4_{n_1}(u) F^3_{n_2}(v) \, du \, dv$$

$$\leqslant c \sum_{a=0}^{\tau-1} \sum_{d=\tau}^{\infty} \frac{2^a (d - \tau + 1) \, 2^\tau}{2^\tau 2^d} \leqslant c \sum_{a=0}^{\tau-1} \frac{2^a}{2^\tau} \leqslant c.$$

Discuss case (13.2). We have to deal with the following subcases:

$$a \leqslant n_1 < b, \qquad \text{then} \quad n_2 \leqslant n_1 + \beta < b + \beta, \tag{13.2.1}$$

$$a < b \leqslant n_1. \tag{13.2.2}$$

First, see (13.2.1). Then, we have (by the formula for $F^4_{n_1}$ in the proof of Lemma 9) $F^4_{n_1}(u) \leqslant c 2^a$ and

$$\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\int_{(I_b(2^{-a-1})\setminus I_{b+1}(2^{-a-1}))\times(I_d\setminus I_{d+1})}\sup_{\substack{\vee\,\mathbf{n}<\tau\\|n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\,F^3_{n_2}(v)\,du\,dv$$

$$\leqslant c\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\frac{1}{2^{b+d}}\sup_{\substack{\vee\,\mathbf{n}<\tau\\|n_1-n_2|\leqslant\beta}} 2^a2^{n_2}(d-n_2+1)$$

$$\leqslant c\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\frac{1}{2^{b+d}}2^a2^{b+\beta}(d-b+1)$$

$$\leqslant c\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\frac{2^a}{2^d}(d-a)^2$$

$$\leqslant c\sum_{d=\tau}^{\infty}\frac{2^\tau}{2^d}(d-\tau)^2\leqslant c.$$

At last, see (13.2.2). By (9.2.3.0) we have $F^4_{n_1}(u)\leqslant c2^a2^{b-n_1}(n_1-b+1)$. This implies

$$\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\int_{(I_b(2^{-a-1})\setminus I_{b+1}(2^{-a-1}))\times(I_d\setminus I_{d+1})}\sup_{\substack{\vee\,\mathbf{n}<\tau\\|n_1-n_2|\leqslant\beta}} F^4_{n_1}(u)\,F^3_{n_2}(v)\,du\,dv$$

$$\leqslant c\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\frac{1}{2^{b+d}}\sup_{\substack{n_1,\,n_2<\tau,\,b\leqslant n_1\\|n_1-n_2|\leqslant\beta}} 2^a2^{b-n_1}2^{n_2}$$

$$\times(d-n_2+1)(n_1-b+1)$$

$$\leqslant c\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\sum_{b=a+1}^{\tau-1}\frac{1}{2^{b+d}}\,2^{a+b}(d-b+c)^2$$

$$\leqslant c\sum_{d=\tau}^{\infty}\sum_{a=0}^{\tau-1}\frac{2^a}{2^d}(d-a+c)^3\leqslant c\sum_{d=\tau}^{\infty}\frac{2^\tau}{2^d}(d-\tau+c)^3\leqslant c.$$

This completes the proof of Lemma 13. ∎

Lemmas 12 and 13 give

Corollary 14. $\int_{(I\setminus I_\tau)\times I_\tau}\sup_{\vee\,\mathbf{n}<\tau,\,|n_1-n_2|\leqslant\beta}F^i_{n_1}(u)\,F^j_{n_2}(v)\,du\,dv\leqslant c$, *where* $i, j=2, 3, 4$.

Lemma 5 and Corollaries 11 and 14 are used to prove the following lemma which is the very base of the proof of Theorem 1.

Define the maximal operator

$$Gf(u, v) := \sup_{\substack{\mathbf{n} \in \mathbf{P}^2 \\ |n_1 - n_2| \leqslant \beta}} G_{n_1, n_2} f(u, v)$$

$$:= \sup_{\substack{\mathbf{n} \in \mathbf{P}^2 \\ |n_1 - n_2| \leqslant \beta}} \left| \int_{I^2} f(x, y)\, G_{n_1}(u \dot{+} x)\, G_{n_2}(v \dot{+} y)\, dx\, dy \right|.$$

Let $f \in L^1(I^2)$, $\operatorname{supp} f \subseteq I_\tau(x^1) \times I_\tau(x^2) =: J_1 \times J_2$, $\int_{J_1 \times J_2} f = 0$ for some $\tau \in \mathbf{N}$, $\mathbf{x} = (x^1, x^2) \in I^2$. Then

LEMMA 15. $\int_{I^2 \setminus (J_1 \times J_2)} Gf \leqslant c\, \|f\|_1$.

Proof.

$$\int_{I^2 \setminus (J_1 \times J_2)} Gf \leqslant \int_{I^2 \setminus (J_1 \times J_2)} \sup_{\substack{\wedge\, \mathbf{n} \geqslant \tau - \beta \\ |n_1 - n_1| \leqslant \beta}} G_{n_1, n_2} f$$

$$+ \int_{I^2 \setminus (J_1 \times J_2)} \sup_{\substack{\vee\, \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} G_{n_1, n_2} f =: A_1 + A_2.$$

By Lemma 5 and the theorem of Fubini we have

$$A_1 \leqslant \int_{I_\tau(x^1) \times I_\tau(x^2)} |f(x, y)| \int_{I^2 \setminus (I_\tau(x^1) \times I_\tau(x^2))} \sup_{\substack{\wedge\, \mathbf{n} \geqslant \tau - \beta \\ |n_1 - n_2| \leqslant \beta}} |G_{n_1}(u \dot{+} x)\, G_{n_2}(v \dot{+} y)|\, du\, dv\, dx\, dy$$

$$= \int_{I_\tau(x^1) \times I_\tau(x^2)} |f(x, y)| \int_{I^2 \setminus (I_\tau \times I_\tau)}$$

$$\times \sup_{\substack{\wedge\, \mathbf{n} \geqslant \tau - \beta \\ |n_1 - n_2| \leqslant \beta}} |G_{n_1}(u)\, G_{n_2}(v)|\, du\, dv\, dx\, dy \leqslant c\, \|f\|_1.$$

On the other hand,

$$A_2 \leqslant \int_{(I \setminus I_\tau(x^1)) \times (I \setminus I_\tau(x^2))} \sup_{\substack{\vee\, \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} G_{n_1, n_2} f + \int_{(I \setminus I_\tau(x^1)) \times I_\tau(x^2)} \sup_{\substack{\vee\, \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} G_{n_1, n_2} f$$

$$+ \int_{I_\tau(x^1) \times (I \setminus I_\tau(x^2))} \sup_{\substack{\vee\, \mathbf{n} < \tau \\ |n_1 - n_2| \leqslant \beta}} G_{n_1, n_2} f =: A_{2,1} + A_{2,2} + A_{2,3}.$$

Since $|G_n| = G_n = \sum_{i=2}^4 F_n^i$, then the theorem of Fubini and Corollary 14 give

$$A_{2,2} \leqslant \int_{I_\tau(x^1)\times I_\tau(x^2)} |f(x,y)| \int_{(I\setminus I_\tau)\times I_\tau} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} |G_{n_1}(u)\, G_{n_2}(v)|\, du\, dv\, dx\, dy$$

$$\leqslant \sum_{i,j=2}^{4} \int_{I_\tau(x^1)\times I_\tau(x^2)} |f(x,y)|$$

$$\times \int_{(I\setminus I_\tau)\times I_\tau} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} F_{n_1}^i(u)\, F_{n_2}^j(v)\, du\, dv\, dx\, dy \leqslant c\, \|f\|_1.$$

In the same way we have $A_{2,3} \leqslant c\,\|f\|_1$.

The rest is a discussion of $A_{2,1}$.

$$A_{2,1} \leqslant \int_{(I\setminus I_\tau(x^2))\times(I\setminus I_\tau(x^2))} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} \Bigg| \int_{I_\tau(x^1)\times I_\tau(x^2)} f(x,y)$$

$$\times (F_{n_1,\tau}^2(u\dot{+}x) + F_{n_1}^3(u\dot{+}x) + F_{n_1}^4(u\dot{+}x))$$

$$\times (F_{n_2,\tau}^2(v\dot{+}y) + F_{n_2}^3(v\dot{+}y) + F_{n_2}^4(v\dot{+}y))\, dx\, dy \Bigg|\, du\, dv$$

$$+ \int_{(I\setminus I_\tau(x^1))\times(I\setminus I_\tau(x^2))} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_1-n_2|\leqslant\beta}} \Bigg| \int_{I_\tau(x^1)\times I_\tau(x^2)} f(x,y)$$

$$\times \Bigg(F_{n_1}^{2,\tau}(u\dot{+}x) \sum_{j=2,3,4} F_{n_2}^j(v\dot{+}y)$$

$$+ \sum_{i=2,3,4} F_{n_1}^i(u\dot{+}x)\, F_{n_2}^{2,\tau}(v\dot{+}y) \Bigg)\, dx\, dy \Bigg|\, du\, dv =: A_{2,1,1} + A_{2,1,2}.$$

Since for $(x,y) \in I_\tau(x^1)) \times I_\tau(x^2))$, $(u,v) \in (I\setminus I_\tau(x^1)) \times (I\setminus I_\tau(x^2))$ then $(u\dot{+}x, v\dot{+}y) \in (I\setminus I_\tau)\times(I\setminus I_\tau)$. Thus by (10) we have on this set $F_{n_1}^3 = F_{n_1,\tau}^3$. It follows that $F_{n_1,\tau}^2(u\dot{+}x)$, $F_{n_1}^3(u\dot{+}x)$, $F_{n_1}^4(u\dot{+}x)$, $F_{n_2,\tau}^2(v\dot{+}y)$, $F_{n_2}^3(v\dot{+}y)$, and $F_{n_2}^4(v\dot{+}y)$ are constants as (x,y) ranges over the set $I_\tau(x^1)\times I_\tau(x^2)$. Consequently,

$$A_{2,1,1} = \int_{(I\setminus I_\tau(x^1))\times(I\setminus I_\tau(x^2))} \sup_{\substack{\vee \mathbf{n}<\tau \\ |n_2-n_1|\leqslant\beta}} \Bigg| (F_{n_1,\tau}^2(u\dot{+}x^1) + F_{n_1}^3(u\dot{+}x^1) + F_{n_1}^4(u\dot{+}x^1))$$

$$\times (F_{n_2,\tau}^2(v\dot{+}x^2) + F_{n_2}^3(v\dot{+}x^2) + F_{n_2}^4(v\dot{+}x^2))$$

$$\times \int_{I_\tau(x^1)\times I_\tau(x^2)} f(x,y)\, dx\, dy \Bigg|\, du\, dv = 0.$$

On the other hand, Corollary 11 and the theorem of Fubini give $A_{2,1,2} \leqslant c \|f\|_1$. The proof of Lemma 15 is complete. ∎

After all we turn our attention to Theorem 1. Set

$$\begin{aligned} Tf(u,v) &:= \sup_{\substack{\mathbf{n} \in \mathbf{P}^2 \\ |n_1 - n_2| \leqslant \beta}} T_{n_1, n_2} f(u,v) \\ &= \sup_{\substack{\mathbf{n} \in \mathbf{P}^2 \\ |n_1 - n_2| \leqslant \beta}} \left| \int_{I^2} f(x,y)\, d_{n_1} W(u \dot{+} x)\, d_{n_2} W(v \dot{+} y)\, dx\, dy \right| \\ &= \sup_{\substack{\mathbf{n} \in \mathbf{P}^2 \\ |n_1 - n_2| \leqslant \beta}} |d_{\mathbf{n}}(\mathbf{I}f)(u,v)|. \end{aligned}$$

By the help of Lemma 15 we prove that (operator G and then) operator T is of weak type $(1,1)$. This means that for all $f \in L^1(I^2)$, $\lambda > 0$ the inequality $|Tf > \lambda| \leqslant c \|f\|_1/\lambda$ holds. Since the two-dimensional Walsh polynomials are dense in $L^1(I^2)$ and since for a dyadic polynomial $d_{\mathbf{n}}(\mathbf{I}P) \to P$ (as $\min(n_1, n_2) \to \infty$) where $\hat{P}(n_1, n_2) = 0$ for $n_1 n_2 = 0$ [8], then this implies Theorem 1 by standard argument [7, 10].

In order to this we need a decomposition lemma of type Calderon and Zygmund.

Lemma 16 [5, 10]. *Let* $f \in L^1(I^2)$, $\lambda > \|f\|_1$. *Then* $f = f_0 + \sum_{i=1}^{\infty} f_i$, *where* $\|f_0\|_\infty < 4\lambda$, $\operatorname{supp} f_n \subseteq I_{k_n}(x^1_{(n)}) \times I_{k_n}(x^2_{(n)}) =: J^1_n \times J^2_n$ $(x^1_{(n)}, x^2_{(n)} \in I$, $k_n \in \mathbf{N})$, $\int_{J^1_n \times J^2_n} f_n = 0$, $\|f_n\|_1 \leqslant 8\lambda\, |J^1_n \times J^2_n|$ $(n \in \mathbf{P})$. *The sets* $J^1_n \times J^2_n$ *are disjoint, furthermore* $|\Omega| = |\bigcup_{n \in \mathbf{P}} J^1_n \times J^2_n| \leqslant \|f\|_1/\lambda$.

Proof of Lemma 1. Let $f \in L^1(I^2)$, $\lambda > 0$, apply Lemma 16 and the σ-sublinearity of operator G. Apply also that $\|G_{n_1} \times G_{n_2}\|_1 \leqslant c$ $((n_1, n_2) \in \mathbf{P}^2)$ (this follows from Lemma 4) implies ([9]) that the operator G is of type (∞, ∞), that is, $\|Gf\|_\infty \leqslant c \|f\|_\infty$ for all $f \in L^\infty(I^2)$. Consequently, $|\{Gf_0 > c\lambda\}| = 0$ if $c\lambda > \|f_0\|_\infty$.

$$\begin{aligned} |\{x \in I^2 : Gf > 2c\lambda\}| &\leqslant |\{Gf_0 > c\lambda\}| + |\Omega| + \frac{1}{c\lambda} \int_{I^2 \setminus \Omega} G\left(\sum_{i=1}^{\infty} f_i\right) \\ &\leqslant c\, \frac{\|f\|_1}{\lambda} + \frac{1}{c\lambda} \sum_{i=1}^{\infty} \int_{I^2 \setminus (J^1_i \times J^2_i)} Gf_i \\ &= c\, \frac{\|f\|_1}{\lambda} + \frac{1}{c\lambda} \sum_{j=1}^{\infty} B^j. \end{aligned}$$

Lemma 15 gives

$$B^j \leqslant c \, \|f_j\|_1 \qquad (j \in \mathbf{P}).$$

Consequently,

$$\sum_{j=1}^{\infty} B^j \leqslant c \sum_{j=1}^{\infty} \|f_j\|_1 \leqslant c \, \|f\|_1.$$

That is, operator G is of weak type (1,1). And since

$$|d_n W| \leqslant 12 G_n,$$

then we have

$$|\{x \in I^2 : Tf > \lambda\}| \leqslant |\{x \in I^2 : cG(|f|) > \lambda\}| \leqslant c \, \| \, |f| \, \|_1 = c \, \|f\|_1.$$

This completes the proof of Theorem 1. ∎

2. AN APPLICATION OF TYPE (H, L^1)

Let function a be an H° atom with the corresponding square $I_\tau(x^1) \times I_\tau(x^2) =: J_1 \times J_2 \subset I^2$ $(\tau \in \mathbf{N},\ \mathbf{x} = (x^1, x^2) \in I^2)$. Lemma 15 gives that $\int_{I^2 \setminus (J_1 \times J_2)} Ta \leqslant c \, \|a\|_1 \leqslant c$. That is, operator G is H°-quasi-linear (for the definition of quasi-linearity see [9, 12]). Since the operator $\sup_{n_1, n_2 \in \mathbf{N}} |f * (G_{n_1} \times G_{n_2})|$ is bounded from $L^p(I^2)$ to $L^p(I^2)$ for all $1 < p < \infty$ (see [8]), then so does the operator $Gf = \sup_{|n_1 - n_2| < \beta,\, \mathbf{n} \in \mathbf{N}^2} |f * (G_{n_1} \times G_{n_2})|$. Since G is sublinear and H° is quasi-linear then by the theorem of Weisz [11, Theorem 1] we have

THEOREM 17. *Operator G is of type (H°, L^1), i.e.*

$$\|Gf\|_1 \leqslant c \, \|f\|_{H^\circ} \qquad (f \in H).$$

Since $\|\cdot\|_{H^\circ} \leqslant \|\cdot\|_H$, then this follows that $H \subset H^\circ$. Consequently, we also have

COROLLARY 18. *Operator G is of type (H, L^1).*

Since $Tf \geqslant cG(|f|)$ then we have

COROLLARY 19. *For operator T we have*

$$\|Tf\|_1 \leqslant c \, \| \, |f| \, \|_{H^\circ} \qquad (|f| \in H^\circ).$$

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